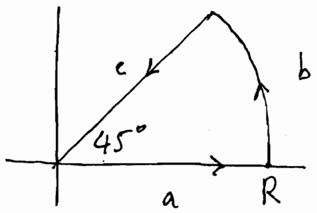
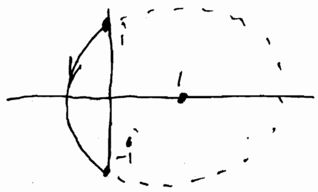
Hoofdstuk 2

2.1 1.



2a. $\varphi(t) = -1+i + t(1-3i-(-1+i)) = -1+i+t(2-4i), \quad 0 \leq t \leq 1$

b. $|z-1| = \sqrt{2}$ van i naar $-i$ via $\operatorname{Re} z \leq 0$



$\varphi(t) = \sqrt{2}\, e^{it}, \quad \frac{3}{4}\pi \leq t \leq \frac{5}{4}\pi.$

3. Kies een $z_0 \in G$; kies een willekeurig ander punt $z \in G$

G is gebied, dus samenhangend en open, dus er is een boog $\gamma(t) \in G$ van z_0 naar z met $a \leq t \leq b$.

Beschouw $g(t) = f(\gamma(t)) = p(t) + i q(t)$, dan zijn f en γ, en dus g, en dus ook p en q continu en differentieerbaar in t, terwijl

$$g'(t) = f'(\gamma(t))\,\gamma'(t) = 0, \text{ zodat } p'=0 \text{ en } q'=0$$

Pas de middelwaardestelling toe op p en q:

$\exists t_1 \in (a,b)$ met $\dfrac{p(b)-p(a)}{b-a} = p'(t_1)$ maar dit is altijd 0, dus $p(a) = p(b)$

$\exists t_2 \in (a,b)$ met $\dfrac{q(b)-q(a)}{b-a} = q'(t_2)$ maar dit is altijd 0, dus $q(a) = q(b)$

Dus $f(z) = f(z_0)$ voor alle z, dus f is constant.

4a. $f(z) = u(x,y)$ en holomorf, dan is $f'(z) = u_x = -i u_y \longrightarrow u_x = 0$

dus $f'(z) = 0$, dus volgt verder zoals opgave 3.

b. $|f(z)|^2 = u^2 + v^2 =$ constant, zodat $u u_x + v v_x = 0$.

$u u_y + v v_y = 0$.

$u * 1^e + v * 2^e = u^2 u_x + uv(v_x + u_y) + v^2 v_y = (u^2+v^2) u_x = 0$ vanwege CR.

$v * 1^e - u * 2^e = vu(u_x - v_y) + v^2 v_x - u^2 u_y = -(u^2+v^2) u_y = 0$ " "

Als $u = v = 0$ dan zijn we klaar. Anders is $u_x = u_y = 0$ zodat $f'(z) = 0$.

Verder volgens opgave 3.

c. Triviaal ? : $f \equiv 1$ op $|z| < 1$, $f \equiv 2$ op $2 < |z| < 3$

2.2

1. $\int_K \bar{z}\, dz = \int_K (x-iy)\, dz$

a. K is $\gamma(t) = t(4+2i)$, dus $x(t) = 4t$, $y(t) = 2t$.

$$\int_K \bar{z}\, dz = \int_0^1 (4t - 2it)(4+2i)\, dt =$$

$$(4+2i)\left[2t^2 - it^2\right]_0^1 = (4+2i)(2-i) = 2(2+i)(2-i) = 10$$

b. $\gamma_1 = 2it$, $\gamma_2 = 2i + t(4+2i-2i) = 2i + 4t$

$$\int_K \bar{z}\, dz = \int_0^1 -2it \cdot 2i\, dt + \int_0^1 (4t - 2i) \cdot 4\, dt$$

$$= 4 \tfrac{1}{2} t^2 \Big|_0^1 + 8\left[t^2 - it\right]_0^1 = 2 + 8 - 8i = 10 - 8i$$

2. $\gamma(t) = it$, $-1 \leq t \leq 1$.

$$\int_K \frac{dz}{z - \sqrt{3}} = \int_{-1}^1 \frac{i\, dt}{it - \sqrt{3}} =$$

$$= \int_{-1}^1 \frac{t - i\sqrt{3}}{t^2+3}\, dt = \int_{-1}^1 \frac{t}{t^2+3} - i\sqrt{3}\int_{-1}^1 \frac{dt}{t^2+3}$$

$$= \frac{1}{2}\log(t^2+3)\Big|_{-1}^1 - i \cdot \arctg \frac{t}{\sqrt{3}}\Big|_{-1}^1 =$$

$$= \frac{1}{2}\log(4) - \frac{1}{2}\log(4) - i \arctg\frac{1}{\sqrt{3}} + i \arctg\frac{-1}{\sqrt{3}} =$$

$$= -2i \cdot \arctg\left(\frac{1}{\sqrt{3}}\right) = -2i \cdot \frac{\pi}{6} = -\frac{1}{3}\pi i$$

3. $\int_K \frac{1}{z}\, dz$

a. $\gamma(t) = \sqrt{2}\, e^{it}$, $-\frac{1}{4}\pi \leq t \leq \frac{1}{4}\pi$

$$\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \frac{\sqrt{2}\, i\, e^{it}\, dt}{\sqrt{2}\, e^{it}} = \int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} i\, dt = it \Big|_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} =$$

$$i \tfrac{1}{4}\pi + \tfrac{1}{4}\pi i = \tfrac{1}{2}\pi i$$

b. $\gamma = 1-i + t(1+i-1+i) = 1-i+2it \quad , \quad 0 \le t \le 1$

$$\int_0^1 \frac{2i\, dt}{1-i+2it} = \int_0^1 \frac{t-\frac{1}{2}+\frac{1}{2}i}{(t-\frac{1}{2})^2+\frac{1}{4}}\, dt =$$

$$\tfrac{1}{2}\log\left((t-\tfrac{1}{2})^2+\tfrac{1}{4}\right)\Big|_0^1 + i \arctg (2t-1)\Big|_0^1 =$$

$$\tfrac{1}{2}\log\left(\tfrac{1}{2}\right) - \tfrac{1}{2}\log\left(\tfrac{1}{2}\right) + i \arctg(1) - i\arctg(-1) =$$

$$= 0 + \frac{\pi}{2} i$$

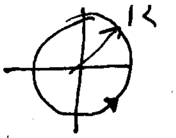
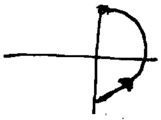
c. $\gamma = \sqrt{2}\, e^{-it} \quad , \quad t = \tfrac{1}{4}\pi \dots \tfrac{7}{4}\pi$

$$\int_{\frac{1}{4}\pi}^{\frac{7}{4}\pi} \frac{-\sqrt{2}\, i e^{-it}}{\sqrt{2}\, e^{-it}}\, dt = \int_{\frac{1}{4}\pi}^{\frac{7}{4}\pi} -i\, dt = -it \Big|_{\frac{1}{4}\pi}^{\frac{7}{4}\pi} = -\tfrac{7}{4}\pi i + \tfrac{1}{4}\pi i = -\tfrac{6}{4}\pi i$$

$$= -\tfrac{3}{2}\pi i$$

4.
$$\left.\begin{aligned} C_1 &= 1+t \\ C_2 &= 2e^{i\pi t} \\ C_3 &= -2+t \\ C_4 &= -e^{-\pi i t} \end{aligned}\right\} \quad t = 0 \dots 1$$

$$\left.\begin{aligned}
\int_{C_1} \tfrac{1}{z}\, dz &= \int_0^1 \frac{1}{1+t}\, dt = \log(1+t)\Big|_0^1 = \log 2 \\
\int_{C_2} \tfrac{1}{z}\, dz &= \int_0^1 \frac{2\pi i\, e^{\pi i t}}{2e^{\pi i t}}\, dt = \pi i t\Big|_0^1 = \pi i \\
\int_{C_3} \tfrac{1}{z}\, dz &= \int_0^1 \frac{dt}{-2+t} = \log(2-t)\Big|_0^1 = \log 0 - \log 2 = -\log 2 \\
\int_{C_4} \tfrac{1}{z}\, dz &= \int_0^1 \frac{+\pi i\, e^{-\pi i t}}{-e^{-\pi i t}}\, dt = -\pi i t\Big|_0^1 = -\pi i
\end{aligned}\right\} \Sigma = 0$$

5. C_R :  $C_R^* =$ 

a. $\left|\int_{C_R}\frac{dz}{z+1}\right| = \left|\int_0^{2\pi}\frac{Rie^{it}}{Re^{it}-1}dt\right| \leq \int_0^{2\pi}\frac{R}{\sqrt{(R\cos t-1)^2+R^2\sin^2 t}}dt$

$= \int_0^{2\pi}\frac{2\pi R}{\sqrt{R^2-2R\cos t+1}}dt \leq \frac{2\pi R}{\sqrt{R^2-2R+1}} = \frac{2\pi R}{R-1}$

b. $\left|\int_{C_R}\frac{dz}{z^2+1}\right| = \left|\int_0^{2\pi}\frac{iRe^{it}}{R^2e^{2it}+1}dt\right| \leq \frac{2\pi R}{\sqrt{(R^2\cos 2t+1)^2+R^2\sin^2 2t}}$

$= \frac{2\pi R}{\sqrt{R^4+2R^2\cos 2t+1}} \leq \frac{2\pi R}{R^2-1} < \varepsilon$ als R groot genoeg.

c. $\left|\int_{C_R^*}\frac{e^{-z}}{(z+1)^2}dz\right| = \left|\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\exp(-Re^{it})iRe^{it}}{(Re^{it}+1)^2}dt\right| \leq \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{e^{-R\cos t}R}{R^2+2R\cos t+1}dt \leq$

$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{R}{R^2-2R+1}dt = \frac{\pi R}{(R-1)^2}$

2.3

1. $f(z) = \frac{1}{z^2+1}$, $z \neq \pm i$

(a). K_1 : $\gamma(t) = \cos t - \frac{1}{2}i\sin t$, $\pi \leq t \leq 2\pi$

$\int_{K_1}\frac{1}{z^2+1}dz = \int_{\pi}^{2\pi}\frac{-\sin t-\frac{1}{2}i\cos t}{(\cos t-\frac{1}{2}i\sin t)^2+1}dt$ = .. moeilijk

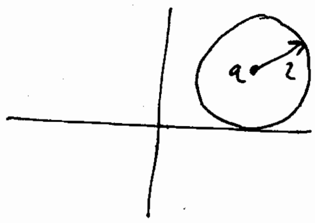
daarom: $\int_{K_1} dz = \int_{-1}^{1}\frac{1}{x^2+1}dx = \operatorname{arctg} x\Big|_{-1}^{1} = \frac{\pi}{2}$.

b $\left|\int_{\substack{|z|=R \\ \operatorname{Im} z \geq 0}} \frac{1}{z^2+1}\,dz\right| = \left|\int_0^{\pi} \frac{iRe^{it}}{R^2e^{2it}+1}\,dt\right| \leq \frac{\pi R}{|1-R^2|} \to 0$ $(R \to 0)$
$R \to \infty$

c. $\int_{-R}^{-1} \frac{dx}{1+x^2} + \int_{K_2} \frac{1}{1+z^2}\,dz + \int_{1}^{R} \frac{dx}{1+x^2} + \int_{|z|=R} \frac{dz}{1+z^2} = 0.$

$\to \int_{K_2} \frac{dz}{1+z^2} = -2\int_1^{\infty} \frac{dx}{1+x^2} = -2 \arctan x \Big|_1^{\infty} = -\frac{1}{2}\pi.$

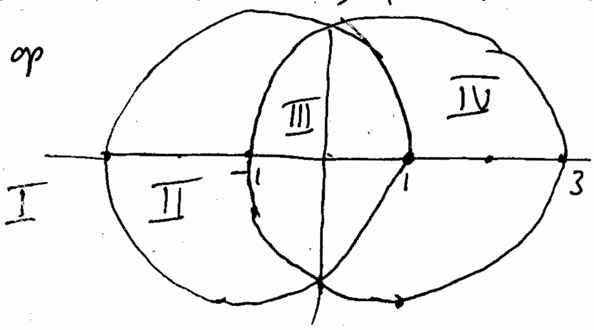
2. $C_a = \{ z \in \mathbb{C} \mid |z-a| = 2 \}$



$f(a) = \int_{C_a} \frac{dz}{z^2-1}$

$= \int_{C_a} \frac{1/2}{z-1} - \frac{1/2}{z+1}\,dz$

Integraal bestaat als $1 \notin C_a$ en $-1 \notin C_a$, $|1-a| \neq 2$ & $|-1-a| \neq 2$
dus voor alle a behalve op



gebied I : $f(a) = 0$
gebied II : $f(a) = -\frac{1}{2} \cdot 2\pi i = -\pi i$
gebied III : $f(a) = \frac{1}{2} \cdot 2\pi i - \frac{1}{2} \cdot 2\pi i = 0$
gebied IV : $f(a) = \frac{1}{2} \cdot 2\pi i = \pi i$

2.4.

1. a) $\frac{e^{z^2}}{z^5} = \frac{1+z^2+\frac{1}{2}z^4+\frac{1}{6}z^6+\cdots}{z^5} \rightarrow Res = \frac{1}{2}$

b) $\frac{e^z}{(z-1)^6} = \frac{e \cdot e^{z-1}}{(z-1)^6} = \frac{e}{(z-1)^6}\left(1+(z-1)+\frac{1}{2}(z-1)^2+\frac{1}{6}(z-1)^3+\ldots\frac{1}{5!}(z-1)^5\ldots\right)$

$\rightarrow Res = \frac{e}{5!}$

c) $\frac{\sinh(\pi z)}{z^4} = \frac{1}{z^4}\left(\pi z+\frac{1}{3!}(\pi z)^3+\ldots\right) \rightarrow Res = \frac{\pi^3}{3!}$

d) $\frac{\sin \pi z}{(z-1)^5} = \frac{\sin(\pi+\pi z-\pi)}{(z-1)^5} = \frac{\sin\pi\cos(\pi z-\pi)+\cos\pi\sin(\pi z-\pi)}{(z-1)^5}$

$= -\frac{1}{(z-1)^5}\left[\pi(z-1)-\frac{1}{3!}\pi^3(z-1)^3+\frac{1}{5!}\pi^5(z-1)^5-\ldots\right]$

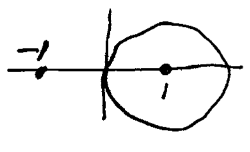
$\rightarrow Res = 0$

2. ~~C geparametriseerd met $\gamma(t)=e^{it}$, $t=0..2\pi$~~

$\int_C \frac{e^{z^2}}{z^2}\,dz = 2\pi i\, Res(0) = 0$

$\int_C \frac{e^{z^2}}{z^5}\,dz = 2\pi i\, Res(0) = \pi i$

3. $\int_C \frac{1}{z^2-1}\,dz = \int_C \frac{1/2}{z-1} - \frac{1/2}{z+1}\,dz$



contour alleen om $z=1$:

$= 2\pi i \cdot Res(1) = 2\pi i \cdot \frac{1}{2} = \pi i$

4. $\frac{1}{2\pi i}\int_K \frac{|\zeta|^2}{\zeta-z}\,dz \qquad |\zeta-1|=1.$

a. $\zeta = 1+e^{it}$: $\bar{\zeta} = 1+e^{-it} = \frac{1+e^{it}}{1+e^{it}-1} = \frac{1}{e^{it}}+1$ ∫.

b. $f(z) = \frac{1}{2\pi i}\int_K \frac{\zeta^2}{(\zeta-1)^2}\frac{1}{\zeta-z}\,d\zeta$

merk op $\frac{\zeta^2}{(\zeta-1)^2} = 1+\frac{2}{\zeta-1}+\frac{1}{(\zeta-1)^2}$

als $z \notin$ binnengebied van K:

$$f(z) = \mathrm{Res}(1) = \frac{2}{1-z}$$

als $z \in$ binnengebied van K, en $z \neq 1$

$$f(z) = \mathrm{Res}(1) + \mathrm{Res}(z) = \frac{2}{1-z} + \frac{z^2}{(z-1)^2} = 1 + \frac{1}{(z-1)^2}$$

als $z = 1$: $f = \mathrm{Res}(1) = 1$ want $\dfrac{z^2}{(z-1)^3} = \dfrac{(z-1)^2 + 2(z-1) + 1}{(z-1)^3}$.

5. $$\int_{|z|=2} \frac{\bar{z}+1}{z^2+1}\,dz$$ (neem: positieve omloopsrichting)

vervang $\bar{z}$ door $\frac{4}{z}$, dan is $\dfrac{\bar{z}+1}{z^2+1} = \dfrac{4+z}{z(z+i)(z-i)}$

integr. $= 2\pi i\,\mathrm{Res}(0) + 2\pi i\,\mathrm{Res}(i) + 2\pi i\,\mathrm{Res}(-i)$

$$\mathrm{Res}(0) = \lim_{z\to 0} z \cdot \frac{4+z}{\cancel{z}(z^2+1)} = 4$$

$$\mathrm{Res}(i) = \lim_{z\to i} (z-i)\frac{4+z}{z(z-i)(z+i)} = \frac{4+i}{i \cdot 2i} = -2 - \tfrac{1}{2}i$$

$$\mathrm{Res}(-i) = \lim_{z\to -i} (z+i)\frac{4+z}{z(z-i)(z+i)} = \frac{4-i}{-i(-2i)} = -2 + \tfrac{1}{2}i$$

totaal: $2\pi i\left(4 - 2 - \tfrac{1}{2}i - 2 + \tfrac{1}{2}i\right) = 0$

6. $$\int_{|z|=1} \frac{2\,\mathrm{Re}(z)}{2z+1}\,dz$$ merk op: $2\,\mathrm{Re}(z) = z + \frac{1}{z}$ op $|z|=1$

$$= \frac{z^2+1}{z}$$

$$= \frac{1}{2}\int_{|z|=1} \frac{z^2+1}{z\left(z+\frac{1}{2}\right)}\,dz = \frac{1}{2}\cdot 2\pi i\left(\mathrm{Res}(0) + \mathrm{Res}(-\tfrac{1}{2})\right)$$

$$= \pi i\left(\frac{0+1}{0+\frac{1}{2}} + \frac{\frac{1}{4}+1}{\frac{1}{4}}\right) = \pi i\,(2 + 1 + 4) = 7\pi i$$

2.5

1. $\frac{1}{2\pi i}\int \frac{c}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\cdot c \cdot 2\pi i \, Res(z) = c$ voor alle $z \in K_{\text{bin}}$

stel $f = c$ op K, dan is $f(z) =$

2. Cauchy is de middelwaardestelling!

$$f(z) = \frac{1}{2\pi i}\int_K \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{met } \zeta(t) = c + Re^{i\theta},\ z = c$$

$$\to f(c) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(c+Re^{i\theta})}{c+Re^{i\theta} - c}\, iRe^{i\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(c+Re^{i\theta})\, d\theta$$

Een harmonische functie is altijd het reële of imaginaire deel van een holomorfe functie, dus neem het reële deel,

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} u(x + R\cos\theta,\ y + R\sin\theta)\, d\theta$$

3. $F(z) = \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}$

a. $|z| < \frac{1}{2} \Leftrightarrow |z^2| < \frac{1}{4} \Leftrightarrow -|z|^2 > -\frac{1}{4} \Leftrightarrow n^2 - |z|^2 > n^2 - \frac{1}{4} > \frac{1}{2}n^2$

zodat

$$\left|\frac{1}{z^2 - n^2}\right| \leq \left|\frac{1}{n^2 - |z|^2}\right| = \frac{1}{n^2 - |z|^2} < \frac{2}{n^2}$$

$\sum \frac{1}{n^2}$ convergeert, dus met Weierstrass is F unif. cvg.

b. Morera + Thm 2.5.12 :

$$w_n = \frac{1}{z^2 - n^2}$$

$\sum w_n$ converge uniformly on $|z| < \frac{1}{2}$

so $F(z) = \sum_1^{\infty} \frac{1}{z^2 - n^2}$ is holomorphic on $|z| < \frac{1}{2}$.

4. $F(z) = \sum_{n=1}^{\infty} \frac{(1-e^{-z})^n}{n}$

a. $\left|\frac{(1-e^{-z})^n}{n}\right| < \frac{(1/2)^n}{n} \leq \left(\frac{1}{2}\right)^n$ zodat met Weierstrass u.c. volgt.

b. Omdat voor $z \in \mathcal{U}$ geldt dat

$$|1-e^{-x-iy}|^2 = 4e^{-x}\left(\sinh\left(\tfrac{1}{2}x\right)^2 + \sin\left(\tfrac{1}{2}y\right)^2\right) \leq K(x^2+y^2)$$

voor e.e.a. $K>0$, is er een goen omgeving van $z=0$ in $\mathcal{U}$. Volgens St 1.3.9, Lem. 2.5.4, St 1.6.5, St 2.5.2 (zie vorige opg.) is F holomorf, dus analytisch. Omdat (vanwege unif. conv.)

$$F'(z) = \sum_{n=1}^{\infty} \frac{n(1-e^{-z})^{n-1}}{n} \cdot e^{-z} = e^{-z}\sum_{n=1}^{\infty}(1-e^{-z})^{n-1}$$

$$= e^{-z}\frac{1}{1-(1-e^{-z})} = 1$$

dus $F(z) = z + \text{const}$. Tenslotte is $F(0) = \sum_{n=1} \frac{0^n}{n} = 0$, dus $F = z$.

Dus de reeks van F bestaat uit 1 term. Dit geldt in geheel $\mathcal{U}$.

5. $F(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{2z}{1-z^2}\right)^{2n+1}$

a. $|1-z^2| \geq |1-|z|^2| > 1-\frac{1}{9} = \frac{8}{9}$.

$$\left|\frac{1}{1-z^2}\right| < \frac{9}{8}$$

$$\left|\frac{2z}{1-z^2}\right| < 2\cdot\frac{1}{3}\cdot\frac{9}{8} = \frac{3}{4}, \text{ dus } \left|\frac{(-1)^n}{2n+1}\left(\frac{2z}{1-z^2}\right)^{2n+1}\right| < \frac{1}{2n+1}\left(\frac{3}{4}\right)^{2n+1}$$

$\sum \frac{1}{2n+1}\left(\frac{3}{4}\right)^{2n+1}$ convergeert, dus F unif. conv.

(b). $$F'(z) = \sum_{n=0}^{\infty}(-1)^n\cdot\left(\frac{2z}{1-z^2}\right)^{2n}\cdot\frac{2+2z^2}{(1-z^2)^2}$$

$$= \frac{2+2z^2}{(1-z^2)^2}\sum_{n=0}^{\infty}\left(\frac{-4z^2}{(1-z^2)^2}\right)^n = \frac{2+2z^2}{(1-z^2)^2}\cdot\frac{1}{1+\frac{4z^2}{(1-z^2)^2}} = \frac{2+2z^2}{(1+z^2)^2}$$

$$= \frac{2}{1+z^2}$$

6. $f(z) = \sum_{n=1}^{\infty} \frac{e^{-nz}}{n}$.

a. f is convergent waar $|e^{-z}| < 1$.

$|e^{-x-iy}| = e^{-x} < 1$, dus $x = Re(z) > 0$.

b. $f'(z) = \sum_{n=1}^{\infty} \frac{-n e^{-nz}}{n} = -\sum_{n=1}^{\infty} e^{-nz} =$

$= -\frac{e^{-z}}{1-e^{-z}} = -\frac{1}{e^{z}-1} = \frac{1}{1-e^{z}}$

7. $\sum_{n=0}^{\infty} \frac{\sin(nz)}{n!} = \frac{1}{2i} \sum_{n=0}^{\infty} \left(\frac{1}{n!} e^{inz} - \frac{1}{n!} e^{-inz} \right)$.

$\left| \frac{e^{inz}}{n!} \right| = \left| \frac{e^{inx} \cdot e^{-ny}}{n!} \right| = \frac{e^{-ny}}{n!}$ terwijl $\sum_{n=0}^{\infty} \frac{e^{-ny}}{n!}$ convergent voor alle y

$\left| \frac{e^{-inz}}{n!} \right| = \frac{e^{ny}}{n!}$ idem.

dus $\sum \frac{1}{n!} e^{\pm inz}$ convergent voor alle z, dus som ook.

Som is: $\sum_{n=0}^{\infty} \frac{e^{\pm inz}}{n!} = \exp(e^{\pm iz}) = \exp(\cos z \pm i \sin z)$

dus

$\frac{1}{2i} \left(\exp(\cos z + i \sin z) - \exp(\cos z - i \sin z) \right)$

$= \exp(\cos z) . \sin(\sin z)$.

8. a $f(z) = \frac{1}{z-1} = \frac{1}{z-a+a-1} = \frac{1}{a-1} \cdot \frac{1}{\frac{z-a}{a-1}+1} = \frac{-1}{1-a} \cdot \frac{1}{1-\frac{z-a}{1-a}} =$

$= -\frac{1}{1-a} \sum_{n=0}^{\infty} \frac{(z-a)^n}{(1-a)^n}$

$= \frac{1}{2} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{(z-3)^n}{2^n}$

b. $f(z) = \cosh(z)$, $a = \frac{1}{2}\pi i$

omdat $\cosh(x+y) = \mathrm{ch}\, x\, \mathrm{ch}\, y + \mathrm{sh}\, x\, \mathrm{sh}\, y$

is

$$\cosh(z) = \cosh(z - \tfrac{\pi}{2}i + \tfrac{\pi}{2}i) = \cosh(z - \tfrac{\pi}{2}i)\,\mathrm{ch}(\tfrac{\pi}{2}i) + \sinh(z - \tfrac{\pi}{2}i)\sinh(i\tfrac{\pi}{2})$$

$$= \mathrm{ch}(z - \tfrac{\pi}{2}i)\cos(\tfrac{\pi}{2}) + i\,\mathrm{sh}(z - \tfrac{\pi}{2}i)\sin\tfrac{\pi}{2} = i\sinh(z - \tfrac{\pi}{2}i)$$

$$= i\sum_{n=0}^{\infty} \frac{(z - \frac{1}{2}\pi i)^{2n+1}}{(2n+1)!}$$

c. $f(z) = e^z + \frac{1}{(1-z)^2}$, $a = 0$.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n. \rightarrow + \frac{1}{(1-z)^2} = \sum_{n=1} n z^{n-1} = \sum_{n=0}^{\infty} (n+1) z^n$$

$$e^z + \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}\left(\frac{1}{n!} + n+1\right) z^n, \text{ geldig voor } |z| < 1.$$

d. $f(z) = \frac{1}{(1-z)(1+z^2)}$, $a = 0$.

$$= \frac{1/2}{1-z} + \frac{1}{4}i\,\frac{z+1}{z+i} - \frac{1}{4}i\,\frac{z+1}{z-i} = \frac{\frac{1}{2}}{1-z} + \frac{1}{4}\,\frac{1+z}{1-iz} + \frac{1}{4}\,\frac{1+z}{1+iz}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} z^n + \frac{1}{4}(1+z)\sum_{n=0}(iz)^n + \frac{1}{4}(1+z)\sum_{n=0}(-iz)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} z^n + \frac{1}{4}\sum_{n=0}^{\infty} i^n z^n + \frac{1}{4}\sum_{n=0} i^n z^{n+1} + \frac{1}{4}\sum_{0}(-i)^n z^n + \frac{1}{4}\sum_{0}(-i)^n z^{n+1}$$

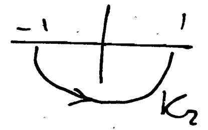
$$= \frac{1}{2}\sum_{n=0} z^n + \frac{1}{4}\sum_{n=0}\left(i^n z^n + (-i)^n z^n\right) + \frac{1}{4}\sum_{n=1} -i\,i^n z^n + \frac{1}{4}\sum_{n=1} i(-i)^n z^n.$$

$$= \frac{1}{2} + \frac{1}{4}(1+1) + \sum_{n=1}^{\infty}\left(\frac{1}{2} + \frac{1}{4}(1-i)i^n + \frac{1}{4}(1+i)(-i)^n\right) z^n$$

$$= 1 + \frac{1}{2}\sum_{n=1}^{\infty}\left(1 + \sqrt{2}\cos\left(\tfrac{1}{2}\pi n - \tfrac{1}{4}\pi\right)\right) z^n.$$

korter: $f = \frac{1+z}{1-z^4} = (1+z)\sum_{n=0}^{\infty} z^{4n} = \sum_{n=0}^{\infty}\left(z^{4n} + z^{4n+1}\right)$

9. $K_1 = \{ z \in \mathbb{C} \mid |z| = 1, \ \operatorname{Im} z \geq 0 \}$, 

$K_2 = \{ z \in \mathbb{C} \mid |z| = 1, \ \operatorname{Im} z \leq 0 \}$ 

$K_1 : \ z = e^{\pi i - \pi i t}, \ 0 \leq t \leq 1$

$K_2 : \ z = e^{-\pi i + \pi i t}, \ 0 \leq t \leq 1$.

$$f_m(z) = \int_{K_m} \frac{\zeta}{\zeta - z} \, d\zeta.$$

a. Kort antwoord: volgens Lemma 2.5.4 is f_m in een machtreeks te ontwikkelen, dus analytisch, dus holomorf.

Lang antwoord:

$$\frac{f_m(z+h) - f(z)}{h} = \frac{1}{h} \int_{K_m} \frac{\zeta}{\zeta - z - h} - \frac{\zeta}{\zeta - z} \, d\zeta =$$

$$= \frac{1}{h} \int \frac{\zeta h}{(\zeta - z)(\zeta - z - h)} \, d\zeta = \int \frac{\zeta}{(\zeta - z)(\zeta - z - h)} \, d\zeta \to \int \frac{\zeta \, d\zeta}{(\zeta - z)^2}$$

b. $$f_1'(z) = \int_{K_1} \frac{\zeta}{(\zeta - z)^2} \, d\zeta.$$

$$f_1''(z) = 2 \int_{K_1} \frac{\zeta}{(\zeta - z)^3} \, d\zeta.$$

etc.: $$f_1^{(n)}(z) = n! \int_{K_1} \frac{\zeta}{(\zeta - z)^{n+1}} \, d\zeta.$$

$$f_1'(0) = \int_{K_1} \frac{\zeta}{\zeta^2} \, d\zeta = \int_{K_1} \frac{1}{\zeta} \, d\zeta =$$

$$\int_0^1 \frac{e^{\pi i - \pi i t} \cdot -\pi i}{e^{\pi i - \pi i t}} \, dt = -\pi i t \Big|_0^1 = -\pi i$$

c. $f_1(z) - f_2(z) = -\int_{-K_1} \frac{\zeta}{\zeta - z} d\zeta - \int_{K_2} \frac{\zeta}{\zeta - z} dz =$

$$= -\int_{|\zeta|=1} \frac{\zeta}{\zeta - z} d\zeta = -2\pi i \cdot 1 \quad \text{als } |z| < 1$$

$$= 0 \quad \text{als } |z| > 1.$$

= niet gedefinieerd als $|z| = 1$.

d. $f_1^{(n)}(0) = n! \int_{K_1} \frac{\zeta}{\zeta^{n+1}} d\zeta = n! \int_{K_1} \frac{1}{\zeta^n} d\zeta = n! \int_0^1 e^{\pi i - \pi i t} \cdot -\pi i \left(e^{\pi i - \pi i t}\right)^{-n} dt$

$$= -\pi i\, n!\, e^{\pi i - n\pi i} \cdot \int_0^1 e^{(n-1)\pi i t} dt$$

$$= +\cancel{\pi i}\, n! (-1)^n \cdot \frac{e^{(n-1)\pi i t}}{(n-1)\cancel{\pi i}} \Big|_0^1 = \frac{n!}{n-1} (-1)^n \left(e^{(n-1)\pi i} - 1\right)$$

$$= \frac{n!}{n-1} \left(\cancel{(-1)^n} \cdot -1 \cdot \cancel{(-1)^n} - (-1)^n\right) = \frac{-n!}{n-1} \left(1 + (-1)^n\right).$$

$f_2^{(n)}(0)$ idem.

Ook: $\frac{\zeta}{\zeta - z} = \frac{1}{1 - \left(\frac{z}{\zeta}\right)} = \sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n$

$$\int \frac{\zeta}{\zeta - z} d\zeta = \sum \int_{K_1} \left(\frac{z}{\zeta}\right)^n d\zeta = \sum_{n=0} z^n \int_{K_1} \zeta^{-n} d\zeta.$$

conv. straal is 1.

$$= \sum_{n=0}^{\infty} z^n \int_0^1 e^{-n\pi i + n\pi i t} \cdot -\pi i e^{\pi i - \pi i t} dt$$

$$= 2 - \pi i z + \sum_{n=2}^{\infty} \frac{1 + (-1)^n}{n-1} z^n.$$

ok.

10. $\frac{z}{\sin z} = \frac{1}{1-\frac{1}{6}z^2+\frac{1}{5!}z^4+\dots} = c_0 + c_2 z^2 + c_4 z^4 + \dots$

$1 = c_0 + z^2(c_2 - \frac{1}{6}c_0) + z^4(c_4 - \frac{1}{6}c_2 + \frac{1}{5!}c_0) + \dots$

$c_0 = 1$, $c_2 = \frac{1}{6}$, $c_4 = -\frac{1}{5!} + \frac{1}{6}\cdot\frac{1}{6} = \frac{10}{360} - \frac{3}{2.3.4.5.3.10}$
$= \frac{7}{360}$

$\tan z = \frac{\sin z}{\cos z} = \frac{z(1-\frac{1}{6}z^2+\frac{1}{5!}z^4)}{1-\frac{1}{2}z^2+\frac{1}{4!}z^4} = c_0 + c_1 z + c_2 z^2 + c_3 z^3$
$= z.(c_1 + c_3 z^2 + \dots)$

$1-\frac{1}{6}z^2+\frac{1}{5!}z^4 = c_1 + z^2(c_3 - \frac{1}{2}c_1) + z^4(c_5 - \frac{1}{2}c_3 + \frac{1}{4!}c_1)$

$c_1 = 1$, $-\frac{1}{6} = c_3 - \frac{1}{2}.1 \rightarrow c_3 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$.

$c_5 - \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{24}.1 = \frac{1}{120}. \rightarrow c_5 = \frac{1}{6} - \frac{1}{24} + \frac{1}{120} = \frac{20-5+1}{120} = \frac{16}{120}$
$= \frac{2}{15}$

11. a. dichtstbijzijnde singulariteit:

$e^{\pi z}+1=0$, $e^{\pi z}=-1=e^{\pi i+2k\pi i}$ behalve $z=\pm i$ want $ch\pi - ch\pi = 0$

$z = i + 2ki = -3i, \cancel{-i}, \cancel{i}, 3i, \dots$

dus $z=3i$, dus $R=3$.

b. dichtstbijzijnde singulariteit: $z=1$, dus $R=1$.

12. $f(z) = (z-a)^k g(z)$. g analytisch

$f'(z) = k(z-a)^{k-1}g(z) + (z-a)^k g'(z)$

$\frac{f'(z)}{f(z)} = \frac{k(z-a)^{k-1}\cancel{g(z)}}{(z-a)^k \cancel{g(z)}} + \frac{\cancel{(z-a)^k} g'(z)}{\cancel{(z-a)^k} g(z)}$

$= \frac{k}{z-a} + \frac{g'(z)}{g(z)}$

$\frac{1}{2\pi i}\int_K \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int \frac{k}{z-a}dz + \frac{1}{2\pi i}\int \frac{g'}{g}dz$

$= k$

2.6 **1 a** $\frac{1}{e^z} = e^{-z}$ is geheel.

b. $e^{\frac{1}{z}}$ is niet geheel ($f(0)$ niet gedefinieerd)

c. $\frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^{2n} \frac{z^{2n}}{(2n+1)!}$ is geheel.

d. $\frac{\sqrt{3}\, z \cos z - \sin(z\sqrt{3})}{z^5} = \frac{\sqrt{3}\, z\left(1 - \frac{1}{2} z^2 + \frac{1}{4!} z^4\right) - \left(z\sqrt{3} - \frac{1}{3!} z^3 3\sqrt{3} + \frac{1}{5!} z^5 3^{\frac{5}{2}}\right)}{z^5}$

$= \frac{\cancel{\sqrt{3}\, z} - \cancel{\frac{1}{2}\sqrt{3}\, z^3} + \frac{1}{24}\sqrt{3}\, z^5 - \cancel{\sqrt{3}\, z} + \cancel{\frac{1}{6} \cdot 3\sqrt{3}\, z^3} - \frac{1}{5!}\, 9 \sqrt{3}\, z^5 + \dots}{z^5}$

$= \frac{1}{24}\sqrt{3} - \frac{1}{120} 9\sqrt{3} + \dots$ is geheel.

2. $|f(z)| \leq M |z|^n$ voor alle $z \in \mathbb{C}$, f is geheel,

dus $f(z)$ is polynoom : $f(z) = a z^n + P_{n-1}(z)$.
Stel $P_{n-1} \neq 0$, dan is

$\lim_{z \to 0} \frac{f(z)}{z^n} = a + \frac{P_{n-1}(z)}{z^n} \to \infty$ terwijl $\left|\frac{f(z)}{z^n}\right| \leq M$.

dus moet $P_{n-1} \equiv 0$.

Verder is dan $|a z^n| \leq M |z|^n$, dus $|a| \leq M$.

3. f : geheel en $|f(z)| \leq |e^z|$ voor alle $z \in \mathbb{C}$.
Omdat $e^z \neq 0$ is dus

$\frac{f(z)}{e^z}$ geheel en begrensd, dus $f(z) = C e^z$.

4. $f(z)$ is geheel, $\frac{f(z)}{z^2} \to 2$ voor $|z| \to \infty$, $f(0) = 0$, $f'(0) = 1$
Bepaal $f(z)$

f is polynoom $= a z^2 + b z + c$.

$$\left.\begin{aligned} &\frac{f(z)}{z^2} = a + \frac{b}{z} + \frac{c}{z^2} \to a = 2. \\ &f(0) = c = 0 \\ &f'(0) = 2az + b = b = 1 \end{aligned}\right\} f = 2z^2 + z.$$

2.7

1a. $$\frac{13z+9}{(z-3)(z+1)^2} = \frac{3}{z-3} - \frac{3}{z+1} + \frac{1}{(z+1)^2}$$

$$= -\frac{1}{1-\frac{1}{3}z} - \frac{3}{z}\cdot\frac{1}{1+\frac{1}{z}} + \frac{1}{z^2}\,\frac{1}{(1+\frac{1}{z})^2}$$

$$= -\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n z^n - 3\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} + \sum_{n=0}(n+1)\frac{(-1)^n}{z^{n+2}}$$

b. $$\frac{1}{z^2(z-1)} = \cancel{\frac{1}{(z-1+1)^2(z-1)}\,z - \frac{1}{(z+z-1)^2} - \frac{1}{1+z-1} + \frac{1}{z-1}}$$

$$= -\frac{1}{z^2} - \frac{1}{z} + \frac{1}{z-1} = -\frac{1}{(y-1)^2} - \frac{1}{y-1} + \frac{1}{y-2} \quad \text{met } y = z+1$$

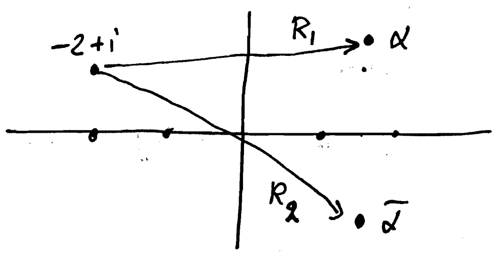
$$= -\frac{1}{y^2}\,\frac{1}{(1-\frac{1}{y})^2} - \frac{1}{y}\,\frac{1}{1-\frac{1}{y}} - \frac{1}{2}\,\frac{1}{1-\frac{1}{2}y}$$

$$= -\sum_{n=0}^{\infty}(n+1)y^{-n-2} - \sum_{n=0}^{\infty} y^{-n-1} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n y^n$$

2a. $$f(z) = \frac{2z-3}{z^2-3z+4}, \quad \text{rond } z = -2+i$$

niet de bedoeling! daardoor heel ingewikkeld.

$$f(z) = \frac{2z-3}{i\sqrt{7}}\left(\frac{1}{z-\alpha} - \frac{1}{z-\bar{\alpha}}\right) \quad \text{met } \alpha = \frac{3}{2} + i\frac{1}{2}\sqrt{7}.$$



$R_1 = \sqrt{15-\sqrt{7}} \approx 3.515$

$R_2 = \sqrt{15+\sqrt{7}} \approx 4.201$

convergent op $R_1 < |z+2-i| < R_2$

Stel $y = z+2-i$

dan

$$f(z) = \frac{2z-3}{i\sqrt{7}}\left[\frac{1}{y\left(1-\frac{\beta}{y}\right)} - \frac{1}{\gamma\left(1-\frac{y}{\gamma}\right)}\right] \quad \text{met} \quad \beta = \tfrac{7}{2} + \left(\tfrac{1}{2}\sqrt{7}-1\right)i, \quad \gamma = \tfrac{7}{2} - \left(\tfrac{1}{2}\sqrt{7}+1\right)i$$

$$\frac{1}{1-\frac{\beta}{y}} = \sum_{n=0}^{\infty}\left(\frac{\beta}{y}\right)^n \qquad \frac{1}{1-\frac{y}{\gamma}} = \sum_{n=0}^{\infty}\left(\frac{y}{\gamma}\right)^n$$

etc.

3. $f(z) = e^z + \dfrac{1}{(z-1)^2}$

a. $$e^z + \frac{1}{(z-1)^2} = e^z + \frac{1}{z^2\left(1-\frac{1}{z}\right)^2} = e^z + \frac{1}{z^2}\sum_{n=0}^{\infty}(n+1)\, z^{-n} \quad \text{conv. voor } |z|>1.$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}z^n + \sum_{n=0}^{\infty}(n+1)\, z^{-n-2}$$

b. $$e^z + \frac{1}{(z-1)^2} = e^{z-1+1} + \frac{1}{(z-1)^2} = \frac{1}{(z-1)^2} + e\cdot\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

4. $$f(z) = \sum_{n=1}^{\infty}(z-1)^{-2n} + \sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

$$= \frac{1}{(z-1)^2}\cdot\frac{1}{1-\frac{1}{(z-1)^2}} + e^{z-1} = \frac{1}{(z-1)^2-1} + e^{z-1} = \frac{1}{z^2-2z} + e^{z-1}$$

$$= -\frac{1}{2}\cdot\frac{1}{z}\cdot\frac{1}{1-\frac{1}{2}z} + e^{z-1} = -\sum_{n=0}\left(\frac{1}{2}\right)^{n+1} z^{n-1} + e^{-1}\cdot\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

5. a. $\operatorname{tg} z$ is analytisch op $\frac{1}{2}\pi < |z| < \frac{3}{2}\pi$

terwijl $\operatorname{tg} z \simeq z + \frac{1}{3}z^3 + \dots$

zodat

$$c_0 = \frac{1}{2\pi i}\int_K \frac{\operatorname{tg} z}{z^2}\,dz = \frac{1}{2\pi i}\int \frac{z}{z^2} + \dots\, dz = 1$$

$$c_1 = \frac{1}{2\pi i}\int_K \frac{z}{z^3} + \frac{1}{3}\frac{z^3}{z^3}\dots\, dz = 0$$

$$c_n = \frac{1}{2\pi i}\int_K \operatorname{tg} z \cdot z^{+n+1}\, dz = 0 \quad \text{voor } n \geq 1.$$

b. $\dfrac{1}{e^z-1}$ is analytisch op $2\pi < |z| < 4\pi$ omdat $e^z = 1$ als $z = 0, \pm 2\pi i, \pm 4\pi i$.

$$(e^z-1)^{-1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z + \dots, \text{ zodat } c_0 = \frac{1}{2\pi i}\int \frac{(e^z-1)^{-1}}{z}\,dz = \frac{1}{2\pi i}\int \frac{1}{z^2} - \frac{1}{2z} + \dots\, dz = -\frac{1}{2}$$

$$c_1 = \frac{1}{2\pi i}\int \frac{1}{z^3} - \frac{1}{2z^2} + \frac{1}{12z} + \dots\, dz = \frac{1}{12}, \quad c_{-1} = \frac{1}{2\pi i}\int \frac{1}{z} - \frac{1}{2} + \dots\, dz = 1$$

6 a. $\frac{z}{\sin z} = -\frac{z}{\sin(z-\pi)} \approx -\frac{\pi}{z-\pi}$, dus $a=-\pi$

$\frac{z}{\sin z} = -\frac{z}{\sin(z+\pi)} \approx -\frac{-\pi}{z+\pi} = \frac{\pi}{z+\pi}$, dus $b=\pi$

b. $\sin z = 0$ in $z = 0, \pm\pi, \pm 2\pi$, etc.

dus $\frac{z}{\sin z}$ is analytisch in $\pi < |z| < 2\pi$

$$c_{-n} = \frac{1}{2\pi i}\int_K \frac{z}{\sin z} \cdot z^{n-1}\, dz.$$

$$= \frac{1}{2\pi i}\int \left(\frac{z}{\sin z} + \frac{\pi}{z-\pi} - \frac{\pi}{z+\pi}\right) z^{n-1}\, dz$$

$$- \frac{1}{2\pi i}\int \left(\frac{\pi}{z-\pi} - \frac{\pi}{z+\pi}\right) z^{n-1}\, dz =$$

$$= 0 - \pi \cdot \pi^{n-1} + \pi(-\pi)^{n-1} = -\pi^n - (-\pi)^n = \begin{cases} -2\pi^n & \text{even} \\ 0 & \text{oneven} \end{cases}$$

zodat

$$\sum_{n=1}^{\infty} c_{-n} z^{-n} = \sum_{n=1}^{\infty} -\pi^n\left(1+(-1)^n\right) z^{-n} = -2\sum_{n=1} \pi^{2n} z^{-2n}$$

$$= -2\sum_{n=1}^{\infty}\left(\frac{\pi}{z}\right)^{2n} = -2\,\frac{\left(\frac{\pi}{z}\right)^2}{1-\frac{\pi^2}{z^2}} = -\frac{2\pi^2}{z^2-\pi^2}$$

10. a. $f(z) = \frac{z e^{\frac{1}{z}} \sin z}{(z-2)^2 (z+\pi)^2}$

$z=2$: 2-voudige pool

$z=-\pi$: 1-voudige pool

$z=0$: essentiele singulariteit.

b. $f(z) = z^2 e^{\frac{1}{z}} - (z^2-2z) \cdot \sin\left(\frac{1}{z-1}\right)$

$z=0$: essentiele singulariteit

$z=1$: essentiele singulariteit.

12c. $\displaystyle\int_{|z|=R} \frac{e^{1/z}}{(1-z)^2}\,dz$

residu in $z=0$; $e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{n}$, $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^{n}$

$$\frac{e^{\frac{1}{z}}}{(1-z)^2} = \left(1+2z+3z^2+4z^3+\dots\right)\left(1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3!z^3}+\dots\right) =$$

$$= \frac{1}{z}\left(1+\frac{2}{2}+\frac{3}{3!}+\frac{4}{4!}+\frac{5}{5!}+\dots\right) + \text{etc.} = \frac{e}{z}+\dots$$

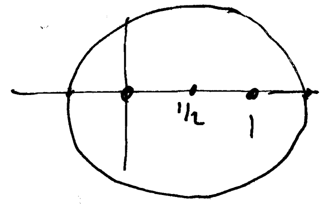
residu in $z=1$: $e^{\frac{1}{z}} = e^{\frac{1}{1-(1-z)}} = e^{1+(1-z)+(1-z)^2\dots} = e\left(1+(1-z)\dots\right)$

$$\frac{e^{1/z}}{(1-z)^2} = \frac{e}{(1-z)^2}\left(1+(1-z)+\dots\right) = \frac{e}{(1-z)^2}+\frac{e}{1-z}+\dots$$

totaal : als $R<1$: $2\pi i e$

als $R>1$: $2\pi i(e-e)=0$

13c. $\displaystyle\int_{|z-\frac{1}{2}|=1} \frac{\cot(\pi z)}{z^2-1}\,dz$



singuliere punten in $z=0$ en $z=1$

$z=0$: $\displaystyle\frac{\cot\pi z}{z^2-1} = \frac{\cos\pi z}{\sin\pi z.(z^2-1)} \sim \frac{1}{-\pi z}$

$z=1$: $\displaystyle\frac{\cot\pi z}{z^2-1} = \frac{\cot(\pi(z-1))}{(z-1)(z+1)} = \frac{1}{(z+1)}\cdot\frac{1}{(z-1)^2}\cdot\frac{1}{\pi}\,\frac{1-\frac{1}{2}\pi^2(z-1)^2}{1-\frac{1}{6}\pi^2(z-1)^2} =$

$$\frac{1}{2}\left(1-\frac{1}{2}(z-1)+\dots\right)\frac{1}{\pi}\,\frac{1}{(z-1)^2}\left(1-\frac{1}{2}\pi^2(z-1)^2\right)\left(1+\frac{1}{6}\pi^2(z-1)^2\right) =$$

$$\frac{1}{2\pi}(z-1)^{-2} - \frac{1}{4\pi}(z-1)^{-1} + \dots$$

totaal : $-2\pi i\frac{1}{\pi} - 2\pi i\cdot\frac{1}{4\pi} = -2i-\frac{1}{2}i = -\frac{5}{2}i$

24. a levert: $f(z) = h(z) + \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{1}{z+1}$, h geheel.

b. levert; $\frac{h}{z} + \frac{A + B(z-1)}{z(z-1)^2} + \frac{1}{z(z+1)} \to \frac{h}{z} \to 1$

dus $h = Pz + Q = z + Q$ ..

c levert: $f(0) = Q + A - B + 1 = 0$

$f'(0) = 1 + 2A - B - 1 = 0$

$f''(0) = 6A - 2B + 2 = 0$

$Q = -2,\ A = -1,\ B = -2.$

25. a. levert: $f(z) = h(z) + \frac{A}{z} + \frac{B}{(z-2)^2} + \frac{5}{z-2}$, h geheel

b levert: $h \to 0$ dus $h \equiv 0$.

c. levert:

$$\left.\begin{aligned} \frac{A}{1} + \frac{B}{1} + \frac{5}{-1} &= 0. \\ \frac{A}{-1} + \frac{B}{9} + \frac{5}{-3} &= 0. \end{aligned}\right\} \quad B = 6,\ A = -1.$$

26. a. $f(z) = h(z) + \frac{A}{z-2} + \frac{B}{z+2}$

b. $\frac{h}{z^3} \to 1$, dus $h = z^3 + Pz^2 + Qz + R$

c. $-z^3 + Pz^2 - Qz + R + \frac{A}{-z-2} + \frac{B}{-z+2} = -z^3 - Pz^2 - Qz - R - \frac{A}{z-2} - \frac{B}{z+2}$

dus $P = 0,\ R = 0,\ A = B$

$f = z^3 + Qz + \frac{A}{z-2} + \frac{A}{z+2}$

d. $f(1) = 1 + Q - A + \frac{1}{3}A = 0$, $Q = \frac{2}{3}A - 1$.

$\int_{|z|=4} f(z)\,dz = 2\pi i \cdot 2A = 6\pi i \to A = \frac{3}{2}$, dus $Q = 0$

2.9

1. $\ln i = \ln e^{\frac{1}{2}\pi i} = \frac{1}{2}\pi i$ (bij hoofdwaarde ln)

$i^i = e^{i \ln i} = e^{i\frac{1}{2}\pi i} = e^{-\frac{1}{2}\pi}$

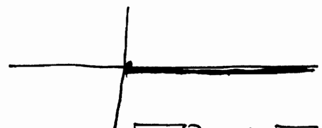
$(\sqrt{i})^{\ln i} = e^{\frac{1}{2}\pi i \cdot \frac{1}{2} \cdot \frac{1}{2}\pi i} = e^{-\frac{1}{8}\pi^2}$

2. $\sqrt{z}\sqrt{w} = \sqrt{zw}$ als $\operatorname{Arg} z + \operatorname{Arg} w = \operatorname{Arg}(zw)$

niet bijv. als $z = i$, $w = -1+i$, $\operatorname{Arg} i = \frac{\pi}{2}$, $\operatorname{Arg}(-1+i) = \frac{3}{4}\pi$, $\operatorname{Arg}(-1-i) = -\frac{3}{4}\pi$

3. Onder de aanname dat met $\sqrt{z}$ de hoofdwaarde wordt bedoeld:

snede van $\sqrt{-z}$ is als $-x < 0$, dus $x > 0$, $y = 0$



$f(1) = \sqrt{-1} = i$, dus $f(x) = \sqrt{-x} = i\sqrt{x}$ als $x > 0$

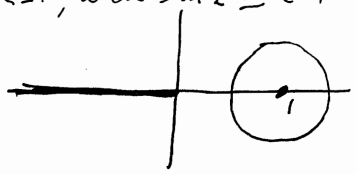
$= \sqrt{-x}$ als $x < 0$.

$f(1+i.0) = \sqrt{-1-i.0} = -i$, dus $f(x+i.0) = -i\sqrt{x}$ (als $x > 0$), anders triv.)

$f(1-i.0) = \sqrt{-1+i.0} = i$, dus $f(x-i.0) = i\sqrt{x}$ (als $x > 0$), anders triv.)

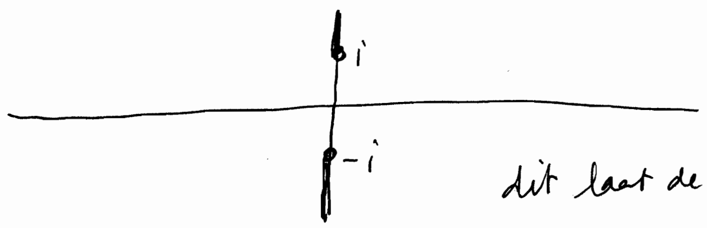
6. $\displaystyle\int_{|z-1|=\frac{1}{2}} \frac{1}{z\ln z}\, dz = 2\pi i$ want analytisch behalve in $z=1$, waar $\ln z \simeq z-1-\frac{1}{2}(z-1)^2+\ldots$

$z = 1 + \frac{1}{2}e^{i\theta}$



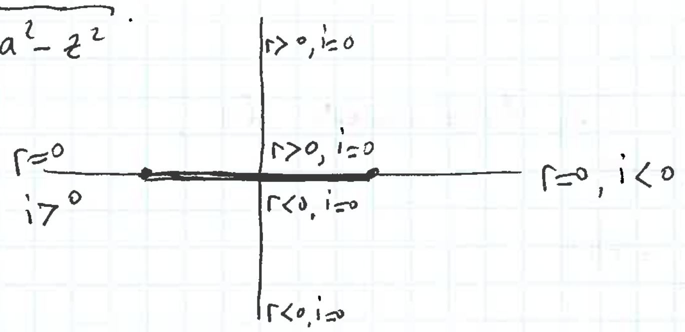
9. $\displaystyle \arctan z = \int_0^z \frac{1}{w^2+1}\, dw = \frac{1}{2i}\int_0^z \frac{1}{w-i} - \frac{1}{w+i}\, dw$

vertakkingssneden vanuit $w = i$ en $w = -i$, bijvoorbeeld



dit laat de reele as ongemoeid.

11. $\sqrt{a^2-z^2}$.



nabij $z=a$:

$a^2-z^2=(a+z)(a-z)\simeq$

$\simeq -2a\varepsilon e^{i\theta} = 2a\varepsilon e^{i(\theta-\pi)}$

$-\pi \leq \theta \leq \pi \to -2\pi \leq \theta-\pi \leq 0$.

$-\pi \leq \frac{\theta-\pi}{2} \leq 0$.

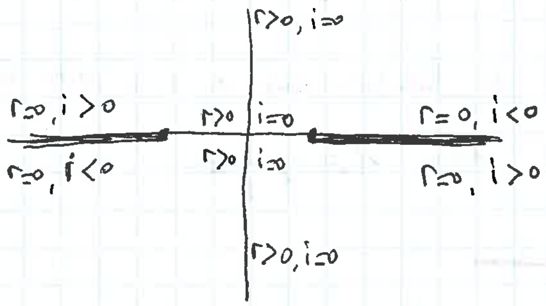
$\theta=\pi$: reel positief

$\theta=-\pi$: reel negatief

$\theta=0$: imag. negatief.

idem bij $z=-a$.

kan ook direct uit explicite formules.

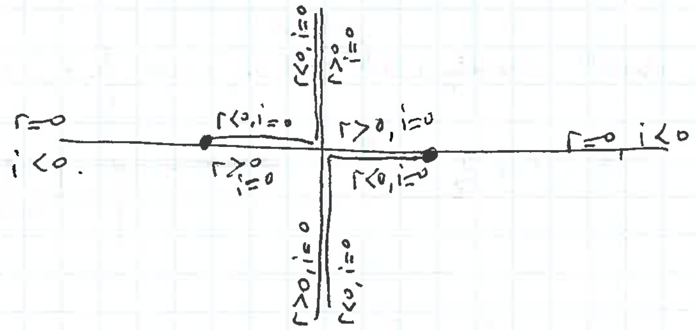


merk op: $a^2-(x+iy)^2=(p+iq)^2$

$\to a^2-x^2+y^2=p^2-q^2$

$xy = -pq$.

dus $p=0$ of $q=0$ alleen als $x=0$ of $y=$



merk op: $\text{Im}\left(\sqrt{a^2-z^2}\right) \leq 0$

12. $\sqrt{\alpha^2-z^2}=p+iq \iff a^2-b^2-x^2+y^2=p^2-q^2$, $xy+ab=-pq$.

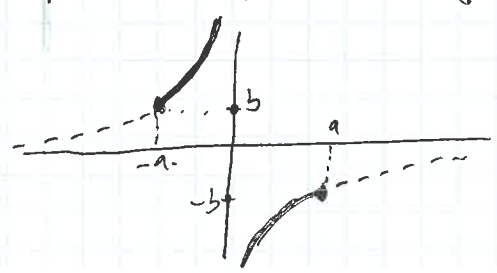
$\text{Im}(f)\leq 0$, dus vertakkingssnede langs de kromme $\text{Im}(f)=q=0$.

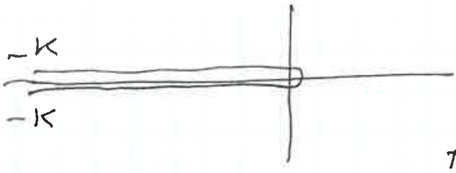
dus $xy=-ab$. (hyperbolen).

echter, niet de hele hyperbool want $p^2=a^2-b^2-x^2+y^2 \geq 0$.

$a^2-b^2-\frac{a^2b^2}{y^2}+y^2 \geq 0 \iff y^4+(a^2-b^2)y^2-a^2b^2=(y^2-b^2)(y^2+a^2)\geq 0$

dus $y^2 \geq b^2$, dus $|y| \geq b$



13. 

Blaas contour op tot cirkel $z = Ke^{i\theta}, -\pi \leq \theta \leq \pi$

$$\frac{1}{2\pi i}\int_\Gamma \lg z\, dz = \frac{1}{2\pi i}\int_{-\pi}^{\pi}(\lg K + i\theta)\, iKe^{i\theta}\, d\theta = \frac{iK}{2\pi i}\left[\lg K \frac{e^{i\theta}}{i} + \theta e^{i\theta} + ie^{i\theta}\right]_{-\pi}^{\pi}$$

$$= \frac{K}{2\pi}\left[0 - 2\pi + 0\right] = -K.$$

18. Check dat $w(z) = z\sqrt{1-\frac{1}{z^2}}_H$ met hoofdwaardewortel $\sqrt{\ }_H$

Dan is voor $|z| > 1$: $\sqrt{1-\frac{1}{z^2}} = 1 - \frac{1}{2}\frac{1}{z^2} - \frac{1}{8}\frac{1}{z^4} + \cdots$

en dus $\left(z^2 - \frac{9}{4}\right) z\sqrt{1-\frac{1}{z^2}} = \left(z^2-\frac{9}{4}\right)\left(z - \frac{1}{2}\frac{1}{z} - \frac{1}{8}\frac{1}{z^3} - \cdots\right) =$

$z^3 - \frac{1}{2}z - \frac{9}{4}z - \frac{1}{8}\frac{1}{z} + \frac{9}{8}\frac{1}{z} + \cdots$

en dus is $\frac{1}{2\pi i}\int_\Gamma \left(z^2 - \frac{9}{4}\right) w(z)\, dz = \frac{1}{2\pi i}\cdot 2\pi i\left(\frac{9}{8} - \frac{1}{8}\right) = 1.$

19. Merk op dat w is opgebouwd uit twee hoofdwaarde-logaritmen.

terwijl: $\lg(z-1) = \lg z + \lg\left(1 - \frac{1}{z}\right)$ (beide HW)

$(-\infty, 0) \cup (0,1)$

$\lg(z+1) = \lg z + \lg\left(1 + \frac{1}{z}\right)$ (beide HW)

$(-\infty, 0)$ $\quad(-1, 0)$.

↳ door de z^{-1} draait het argument terug en heft de snede van $\lg z$ op langs $(-1,0)$

totaal: $w(z) = \lg\left(1 - \frac{1}{z}\right) - \lg\left(1 + \frac{1}{z}\right)$

voor $|z|$ groot: $= -\frac{1}{z} - \frac{1}{2}\frac{1}{z^2} - \frac{1}{3}\frac{1}{z^3} - \cdots - \left(\frac{1}{z} - \frac{1}{2}\frac{1}{z^2} + \frac{1}{3}\frac{1}{z^3} - \cdots\right) =$

$-\frac{2}{z} - \frac{2}{3}\frac{1}{z^3} - \cdots$

zodat $\left(z^2 - \frac{5}{6}\right) w(z) = -2z - \frac{2}{3}\frac{1}{z} + \frac{5\cdot 2}{6z} + \cdots = -2z + \frac{1}{z} \cdots$

en dus $\frac{1}{2\pi i}\int_\Gamma \left(z^2 - \frac{5}{6}\right) w(z)\, dz = 1.$